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Certain Monotonicity Properties of Bessel Functions*

MOURAD E. H. ISMAIL

Department of Mathematics, Arizona State University, Tempe, Arizona 85287

AND

MARTIN E. MULDOON

Department of Mathematics, York University, North York, Ontario, Canada M3J 1P3

Submitted by R. P. Boas

We prove the absolute monotonicity of various expressions involving Bessel functions. These results lead to several Bessel function inequalities which illuminate and generalize recent results of Paris (*SIAM J. Math. Anal.* **15** (1984), 203–205).

1. INTRODUCTION

In [7], Paris proved the inequalities

$$1 \leqq x^{-\nu} J_\nu(\nu x)/J_\nu(\nu) \leqq \exp[\nu(1-x)], \qquad 0 \leqq x \leqq 1, \tag{1.1}$$

when $\nu > 0$. Let $0 < j_{\nu,1} < j_{\nu,2} < \cdots$ be the sequence of positive zeros of $J_\nu(x)$ and let $j'_{\nu,1}$ be the smallest positive zero of $J'_\nu(x)$. A function $f(x)$ is called absolutely monotonic on an interval I if $f^{(n)}(x) \geqq 0$ on I, $n = 0, 1,\ldots$; see Widder [9]. In this note we shall prove the following theorems:

THEOREM 1. *Let $\mu \geqq \nu > -1$, then the function*

$$h(x) := \frac{d}{dx} \ln \left[x^{(\nu-\mu)/2} \exp\left[\frac{x}{4}\left(\frac{1}{\mu+1} - \frac{1}{\nu+1}\right)\right] J_\mu(x^{1/2})/J_\nu(x^{1/2}) \right] \tag{1.2}$$

is absolutely monotonic on $[0, j^2_{\nu,1})$.

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One consequence of Theorem 1 (see Corollary 7 below) is that $x^{-\nu}J_\nu(x)\exp[x^2/(4\nu+4)]$ is decreasing on $(0, j_{\nu,1})$. By contrast, we have the following result:

THEOREM 2. *The function*

$$h_\nu(x) := x^{-\nu}J_\nu(x)\exp(\lambda x), \qquad \lambda := \nu/j'_{\nu,1} \tag{1.3}$$

is a strictly increasing function of x on $[0, j'_{\nu,1}]$ when $\nu > 0$.

THEOREM 3. *Let $\nu > -1$ and define*

$$p(x) := \frac{d}{dx}\ln\left[\frac{J_\nu(ax^{1/2})}{J_\nu(x^{1/2})}\exp\left[\frac{(a^2-1)\,x}{4(\nu+1)}\right]\right], \tag{1.4}$$

then $p(x)$ is absolutely monotonic on $[0, j^2_{\nu,1})$ when $0 < a < 1$.

THEOREM 4. *For $\nu > -1$, the function $f_\nu(x)$ defined by*

$$f_\nu(x^2) = \ln\left[\frac{J_\nu(x)/x^\nu}{j^2_{\nu,1} - x^2}\right] + \frac{(1+\nu)\,x^2}{2j^2_{\nu,1}} \tag{1.5}$$

is increasing on $[0, j^2_{\nu,1})$ and decreasing on $(j^2_{\nu,1}, j^2_{\nu,2})$. Furthermore $-f''_\nu(x)$ is absolutely monotonic on $(0, j^2_{\nu,2})$.

These theorems will be proved in Section 2. The proofs borrow some ideas from [4, 5]. In fact [4, 5] contain similar results involving modified Bessel functions and the gamma function.

Since the exponential of a function having an absolutely monotonic derivative is absolutely monotonic, Theorems 1 and 3, respectively, imply the following useful and possibly more attractive corollaries:

COROLLARY 5. *Assume that $\mu \geq \nu > -1$ and*

$$u_\nu(x^2) = x^{\nu-\mu}\exp\left(\frac{x^2}{4}\left(\frac{1}{\mu+1} - \frac{1}{\nu+1}\right)\right) J_\mu(x)/J_\nu(x). \tag{1.6}$$

Then $u_\nu(x)$ is absolutely monotonic on $[0, j^2_{\nu,1})$.

COROLLARY 6. *If $\nu > -1$ and $0 < a < 1$ then $w_\nu(x)$ is absolutely monotonic on $[0, j^2_{\nu,1})$ where*

$$w_\nu(x^2) = \frac{J_\nu(ax)}{J_\nu(x)}\exp[(a^2-1)\,x^2/(4\nu+4)]. \tag{1.7}$$

We shall see in Section 3 that the limiting case $\mu \to \infty$ of Corollary 5 is the following result:

COROLLARY 7. *For $\nu > -1$ the function*

$$x^{\nu/2} \exp[-x/(4\nu+4)]/J_\nu(x^{1/2}) \tag{1.8}$$

is absolutely monotonic on $[0, j_{\nu,1}^2)$.

Our results imply many inequalities. One example which follows from Corollary 7, as we shall see in Section 3, is

$$J_\nu(\nu y)/J_\nu(\nu) \geq y^\nu \exp[\nu^2(1-y^2)/(4\nu+4)], \tag{1.9}$$

for $y \in (0, 1]$ and $\nu > 0$, with equality if and only if $y = 1$. Observe that (1.9) provides a sharper lower bound than the one given in (1.1). Theorem 2 can be used to sharpen the upper bound in (1.1). The result is Corollary 8:

COROLLARY 8. *When $\nu > 0$ and $x \in [0, 1]$ then*

$$x^{-\nu} J_\nu(\nu x)/J_\nu(\nu) \leqq \exp[\xi(1-x)], \qquad \xi := \nu^2/j'_{\nu,1} \tag{1.10}$$

holds with equality if and only if $x = 1$.

A proof of Corollary 8 will be outlined in Section 3. Note that $\xi < \nu$ (or $\lambda < 1$) because [2, p. 60]

$$j'_{\nu,1} > [\nu(\nu+2)]^{1/2}.$$

2. PROOF OF THEOREMS 1–4

In this section we include proofs of Theorems 1, 2, 3, and 4. Proofs of the remaining results are in Section 3. We start by stating some well-known identities that will be used in the sequel. The Weierstrass factor product for $J_\nu(x)$ is [8, p. 498]

$$2^\nu \Gamma(\nu+1)\, J_\nu(x) = x^\nu \prod_{k=1}^{\infty} [1 - x^2/j_{\nu,k}^2], \qquad \nu > -1. \tag{2.1}$$

The Mittag-Leffler expansion for a quotient of Bessel functions is, see, for example, [2, p. 61],

$$J_{\nu+1}(x)/J_\nu(x) = 2x \sum_{1}^{\infty} (j_{\nu,n}^2 - x^2)^{-1}, \qquad \nu > -1. \tag{2.2}$$

The limiting case $x \to 0$ of (2.2) leads to [8, p. 502]

$$\sum_1^\infty j_{\nu,n}^{-2} = 1/[4(\nu+1)], \qquad \nu > -1. \tag{2.3}$$

Evaluating the residues of the functions in (2.2) at their poles gives, see also [1],

$$\sum_{\substack{n=1\\ n\neq k}}^\infty [j_{\nu,n}^2 - j_{\nu,k}^2]^{-1} = \frac{1}{2}(\nu+1)/j_{\nu,k}^2, \qquad \nu > -1, k = 1, 2, \ldots. \tag{2.4}$$

Proof of Theorem 1. Clearly (2.1) implies

$$\frac{d}{dx}\left[\ln J_\nu(x^{1/2}) - \frac{\nu}{2}\ln x\right] = \sum_1^\infty \{x - j_{\nu,n}^2\}^{-1}, \tag{2.5}$$

which when combined with (1.2) gives

$$h(x) = \frac{1}{4\mu+4} - \frac{1}{4\nu+4} + \sum_1^\infty [(j_{\nu,n}^2 - x)^{-1} - (j_{\mu,n}^2 - x)^{-1}]. \tag{2.6}$$

The zeros $j_{\nu,n}$ are strictly increasing functions of ν, $\nu \in (-1, \infty)$, for fixed n, $n = 1, 2, 3, \ldots$, Watson [8, Sect. 15.8]. This and the representation (2.6) imply the positivity of all the derivatives of $h(x)$ on $[0, j_{\nu,1}^2)$. To see that $h(x)$ is also positive note that $h(x)$ is increasing and $h(0) = 0$, as can be seen from Eq. (2.3).

Proof of Theorem 2. Applying the differential recurrence relation (55) on p. 12 of [2] we get

$$h_\nu'(x)/h_\nu(x) = \lambda - \nu x^{-1} + J_\nu'(x)/J_\nu(x) = \lambda - J_{\nu+1}(x)/J_\nu(x). \tag{2.7}$$

The function $J_{\nu+1}(x)/J_\nu(x)$ is strictly increasing on $[0, j_{\nu,1})$, vanishes at $x = 0$, and tends to $+\infty$ as $x \to j_{\nu,1} - 0$ [6, Lemma 2.5, p. 762]. This shows that $h_\nu'(x)/h_\nu(x)$ is strictly decreasing on $[0, j_{\nu,1})$ and its range is $(-\infty, \lambda]$. From (2.7) we see that

$$h_\nu'(j_{\nu,1}')/h_\nu(j_{\nu,1}') = \lambda - \nu/j_{\nu,1}' = 0.$$

Therefore $h_\nu'(x)/h_\nu(x)$ is positive on $[0, j_{\nu,1}')$ and the proof is complete.

Proof of Theorem 3. The identity (2.5) in the proof of Theorem 1 yields

$$p(x) = \frac{a^2-1}{4\nu+4} + \sum_1^\infty [(j_{\nu,n}^2 - x)^{-1} - (a^{-2}j_{\nu,n}^2 - x)^{-1}]$$

from which one can easily show that $p^{(n)}(x)$ is positive for $n>0$ and $0<x<j_{\nu,1}^2$. This makes $p(x)$ strictly increasing and (2.3) gives $p(0)=0$. Therefore $p(x)$ is positive and the proof is complete.

Proof of Theorem 4. From (1.5) and (2.1) we obtain

$$\frac{d}{dx} f_\nu(x) = \frac{(1+\nu)}{2j_{\nu,1}^2} - \sum_{n=2}^{\infty} (j_{\nu,n}^2 - x)^{-1}. \tag{2.8}$$

Moreover, successive differentiation shows the absolute monotonicity of $-f_\nu''$. In particular f_ν' is decreasing and (2.4) shows that $f'(j_{\nu,1}^2)=0$. Thus f_ν is increasing on $(0, j_{\nu,1}^2)$ and decreasing on $(j_{\nu,1}^2, j_{\nu,2}^2)$.

3. Additional Proofs and Inequalities

The familiar power series representation

$$J_\mu(x) = \sum_0^\infty \frac{(-1)^n (x/2)^{\mu+2n}}{n!\,\Gamma(\mu+n+1)}$$

shows that $(x/2)^{-\mu}\,\Gamma(\mu+1)\,J_\mu(x) \to 1$ as $\mu \to \infty$. This suggests that Corollary 7 is a limiting case of Corollary 5. In fact one can show, as in our proof of Theorem 1, that the logarithm of the function (1.8) is absolutely monotonic. This proves Corollary 7.

The inequality (1.9) follows from Corollary 7 by comparing the values of the function (1.8) at the points $y=\nu^2$ and $y=x^2\nu^2$. Both points belong to $[0, j_{\nu,1}^2)$, since $j_{\nu,1}>\nu$ [8, Sect. 15.3]. Similarly, Corollary 8 states that $h_\nu(\nu x)$ does not exceed $h_\nu(\nu)$, an immediate consequence of Theorem 2.

In the same way, Theorem 4 gives an inequality

$$x^{-\nu}\frac{J_\nu(\nu x)}{J_\nu(\nu)} < \frac{(j_{\nu,1}^2 - \nu^2 x^2)}{(j_{\nu,1}^2 - \nu^2)} \exp[(1+\nu)\,\nu^2(1-x^2)/(2j_{\nu,1}^2)], \tag{3.1}$$

$0<x<1$, $\nu>-1$. This is less elementary than (1.10), but numerical evidence indicates that it is sharper. Also it holds for $\nu>-1$ rather than just $\nu>0$.

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